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# Quadratic-Based Computation of Four-Impulse Optimal Rendezvous near Circular Orbit

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**The well-known problem of minimizing the total characteristic velocity of a spacecraft in an impulsive rendezvous with a satellite in circular orbit is considered by using the Clohessy–Wiltshire equations. It is well known that, for boundary conditions in the plane of the orbit, four impulses at most are required. The mathematical framework is presented for four-impulse optimal rendezvous near a circular orbit resulting in relatively simple formulas that determine if four impulses are required and, if so, how the four optimal velocity increments can be calculated.**

## Introduction

IMPULSIVE rendezvous of a spacecraft in the vicinity of a circular orbit has been and continues to be an appealing area of study. The assumptions of a circular orbit, linearized equations of motion, and instantaneous velocity increments lead to one of the simplest practical mathematical models for analysis and minimization of the characteristic velocity of a spacecraft.

One of the first published papers in this area was by Edelbaum[1] in 1967 using a linearization with respect to the orbital elements. Although this model was also used by Jones[2] in a paper that appeared in 1976, the vast majority of studies use the simpler model found in early papers by Wheelon,[3] Clohessy and Wiltshire,[4] Geyling,[5] Spradlin,[6] and Eggleston.[7] Derivation and applications of the equations of motion that compose this model, generally called the Clohessy–Wiltshire equations, may now be found in books on orbital mechanics.[8–10] Most of the published studies of impulsive rendezvous near a circular orbit use a tool of optimization called primer vector theory, originated by Lawden[11] and formalized by Lion[12] and Lion and Handelsman.[13] This approach has been used successfully by Prussing,[14,15] Jezewski and Donaldson,[16] and Jezewski[17] for the problem of impulsive rendezvous near a circular orbit and has continued into recent times.[18–21]

It was first shown by Neustadt[22] that for linear equations of motion in $n$-dimensional state space, at most $n$ impulses are required for optimization. Simpler proofs of this result have also been found by Stern and Potter,[23] Carter,[24] Carter and Brient,[25] and Prussing.[26] For this reason, an optimal planar rendezvous based on linear equations of motion requires at most four impulses. For the Clohessy–Wiltshire equations, optimal solutions of typical boundary-value problems may require fewer than four impulses.[15] There are problems, however, for which an optimal solution requires exactly four impulses.[14] Numerical methods of solving two-point boundary-value problems encounter more difficulties with four-impulse solutions than with three- or two-impulse solutions. In fact, difficulties tend to increase by an order of magnitude with each increase in the required number of impulses. For this reason, an analytical approach to define all boundary conditions requiring four impulses and a computational procedure to determine the optimal velocity increments are very useful.

Such an approach may be found in a 1969 paper by Prussing.[14] Many important contributions in that paper could be re-examined in the context of an increasing need for rendezvous near a circular orbit, such as those required for assembly, maintenance, and operation of space stations.

Except for the work done by Prussing,[14] it is difficult to find published material on this problem. The present paper repeats Prussing's results as an application of recent new developments in linearized rendezvous[25,27] and focuses on the elimination of numerical problems of convergence for boundary values that require four impulses. A new development of the material is presented with more emphasis on the underlying mathematical ideas, resulting in a simple new quadratic-based procedure [Eqs. (50–53)] for the computation of the impulse times. These lead to the four-dimensional cones[25] that result in an examination of the negativity of the components of a vector $M^{-1}z_f$, where $z_f$ represents the boundary values. This is equivalent to Prussing's $\Delta V_j \geq 0$ in Eq. (12) of his paper.[14]

The primary practical contribution of the present paper is the emphasis on a method for calculation of the velocity impulses. Prussing's paper[14] presented a study of the general problem and obtained solutions. These new approaches allow one to calculate directly a solution for a particular transfer time or for a specified time of a third impulse through a procedure that is based on the quadratic formula.

## Context of the Problem

By the use of an approach of Carter and Brient,[25] this work repeats and extends results found in an important paper by Prussing.[14] For brevity the development in Ref. 25 will not be repeated, but the definitions and symbols used herein are found in that reference.

### Brief Statement of the Problem

From Ref. 25, the linearized $k$-impulse rendezvous problem is the selection of the velocity increments $\Delta \boldsymbol{v}_1, \ldots, \Delta \boldsymbol{v}_k$ and the specific values $\theta_1, \ldots, \theta_k$ of the true anomaly $\theta$ where they are applied on the closed, bounded interval $[\theta_0, \theta_f]$ to minimize the total characteristic velocity

$$J(\Delta v_1, \ldots, \Delta v_k \theta_1, \ldots, \theta_k) = \sum_{i=1}^{k} |\Delta \boldsymbol{v}_i| \tag{1}$$

subject to the constraint

$$\sum_{i=1}^{k} R(\theta_i)\Delta \boldsymbol{v}_i = z_f \tag{2}$$

where

$$R(\theta) = \Phi(\theta)^{-1} B \tag{3}$$

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For linearized rendezvous in a general central force field, the matrix $B$, the state transition matrix $\Phi(\theta)$, and its inverse are obtained from a recent paper.[27]

If the boundary values $z_f$ are in the orbital plane, they can be represented as points in four-dimensional Euclidean space; each $\Delta \boldsymbol{v}_i$ can be represented in two-dimensional space; also it is known[22,23,25,26] that $k = 4$ is sufficient for this problem.

### Necessary and Sufficient Conditions

Necessary conditions for a normal solution of the impulsive rendezvous problem[25] can be obtained by introduction of a Lagrange multiplier $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ and setting equal to zero the partial derivative of the function

$$\sum_{i=1}^{4} |\Delta \boldsymbol{v}_i| + \boldsymbol{\lambda}^T \left[ \sum_{i=1}^{4} R(\theta_i)\Delta \boldsymbol{v}_i - z_f \right] \tag{4}$$

with respect to $\Delta \boldsymbol{v}_1$, $\Delta \boldsymbol{v}_2$, $\Delta \boldsymbol{v}_3$, $\Delta \boldsymbol{v}_4$, $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_4$. By the using of the notation

$$\alpha_i = |\Delta \boldsymbol{v}_i|, \qquad i = 1, 2, 3, 4 \tag{5}$$

these necessary conditions, along with Eq. (2), can be stated in terms of the primer vector

$$\boldsymbol{p}(\theta) = R(\theta)^T \boldsymbol{\lambda} \tag{6}$$

as follows:

$$\Delta \boldsymbol{v}_i = 0, \quad \text{or} \quad \Delta \boldsymbol{v}_i = -\boldsymbol{p}(\theta_i)\alpha_i, \qquad i = 1, 2, 3, 4 \tag{7}$$

$$\sum_{i=1}^{4} R(\theta_i)\boldsymbol{p}(\theta_i)\alpha_i = -z_f \tag{8}$$

$$\alpha_i \ge 0, \qquad i = 1, 2, 3, 4 \tag{9}$$

$$\Delta \boldsymbol{v}_i = 0, \quad \text{or} \quad |\boldsymbol{p}(\theta_i)| = 1, \qquad i = 1, 2, 3, 4 \tag{10}$$

$$\theta_i = \theta_0, \quad \text{or} \quad |\boldsymbol{p}(\theta_i)|' = 0, \quad \text{or} \quad \theta_i = \theta_f \qquad i = 1, 2, 3, 4 \tag{11}$$

The work of Prussing and Clifton[20] and Prussing[26] shows that these necessary conditions, strengthened by the additional necessary condition

$$|\boldsymbol{p}(\theta)| \le 1, \qquad \theta_0 \le \theta \le \theta_f \tag{12}$$

are sufficient for a normal optimal solution of the linear impulsive rendezvous problem. The use of the geometric necessary conditions (10–12) in the analysis of linear and nonlinear problems goes back to at least the 1960s[11–13] and has come to be known as primer vector theory.

### Application to Rendezvous near a Circular Orbit

Application of the necessary and sufficient conditions (7–12) requires, as input, the boundary values $z_f$ and the matrix $R(\theta)$ that is obtained from Eq. (3). For the planar problem of rendezvous near a circular orbit, as in other problems, the fundamental matrix $\Phi(\theta)$ and even the matrix $B$ can be presented in various forms. In this paper the forms will be taken from Ref. 27 as

$$\Phi(\theta) = \begin{bmatrix} -2\cos(\theta) & -2\sin(\theta) & -3\theta & 1 \\ 2\sin(\theta) & -2\cos(\theta) & -3 & 0 \\ \sin(\theta) & -\cos(\theta) & -2 & 0 \\ \cos(\theta) & \sin(\theta) & 0 & 0 \end{bmatrix} \tag{13}$$

$$\Phi(\theta)^{-1} = \begin{bmatrix} 0 & 2\sin(\theta) & -3\sin(\theta) & \cos(\theta) \\ 0 & -2\cos(\theta) & 3\cos(\theta) & \sin(\theta) \\ 0 & 1 & -2 & 0 \\ 1 & 3\theta & -6\theta & 2 \end{bmatrix} \tag{14}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{15}$$

The form of the primer vector $\boldsymbol{p}(\theta) = [p_1(\theta), p_2(\theta)]^T$ thus follows from Eqs. (3), (6), and (14),

$$p_1(\theta) = 2\lambda_1 \sin\theta - 2\lambda_2 \cos\theta + \lambda_3 + 3\lambda_4\theta \tag{16a}$$

$$p_2(\theta) = \lambda_1 \cos\theta + \lambda_2 \sin\theta + 2\lambda_4 \tag{16b}$$

Solution of a specific boundary-value problem requires the determination of the numbers $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$.

## Four-Impulse Solutions

Consider the problem of finding all boundary values $z_f$ for which optimal solutions require four impulses on a flight interval $\theta_0 \le \theta \le \theta_f$. The primer vector (16) is subject to the necessary conditions (10–12). Specific values of $\boldsymbol{\lambda}$ must be found satisfying Eq. (12) such that there exist four points $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_4$ on the interval where $|\boldsymbol{p}(\theta_i)| = 1$, $i = 1, \ldots, 4$, and there are at least two interior points where $|\boldsymbol{p}(\theta_i)|' = 0$, $i = 2, 3$.

### Simplification Through a Change of Variable

The number of unknown parameters in Eq. (16) can be reduced from four to three by the following change of variable. Let

$$\lambda_1 = \rho\cos\varphi, \qquad \lambda_2 = \rho\sin\varphi \tag{17}$$

then

$$p_1(\theta) = 2\rho\sin(\theta - \varphi) + \lambda_3 + 3\lambda_4\theta \tag{18a}$$

$$p_2(\theta) = \rho\cos(\theta - \varphi) + 2\lambda_4 \tag{18b}$$

Next introduce the new variable

$$\tau = \theta - \varphi \tag{19}$$

and the new parameters

$$b = \lambda_4/\rho, \qquad c = (\lambda_3 + 3\lambda_4\varphi)/\rho \tag{20}$$

It is assumed that $\rho \ne 0$, otherwise four-impulse solutions cannot occur as shown by Eqs. (18).

By incorporation of these changes, the primer vector can be expressed in terms of $\tau$, obtaining Prussing's formulas,[14]

$$p_1(\tau) = \rho(2\sin\tau + 3b\tau + c) \tag{21a}$$

$$p_2(\tau) = \rho(\cos\tau + 2b) \tag{21b}$$

### Necessary Conditions on the Primer Vector for Four-Impulse Solution

It is seen that the shape is cycloidal, that is, $[p_1(\tau)/2, p_2(\tau)]^T$ is a cycloid. The scale is determined by $\rho$ and the type (or shape) by $b$, and $c$ effects a horizontal translation of the curve. The curve defined the prime vector is periodic having a horizontal period of $2\pi b\rho$ and is symmetric about a vertical axis. Necessary conditions for four-impulse solutions from Eqs. (10) and (11) are

$$\boldsymbol{p}(\tau_i)^T\boldsymbol{p}(\tau_i) = 1, \qquad i = 1, 2, 3, 4 \tag{22}$$

$$\boldsymbol{p}(\tau_i)^T\boldsymbol{p}'(\tau_i) = 0, \qquad i = 2, 3 \tag{23}$$

where $\tau_1$, $\tau_2$, $\tau_3$, and $\tau_4$ are distinct.

Condition (22) asserts that the primer vector must intersect the unit circle at exactly four distinct points. This will be called the intersection condition. Condition (23) asserts that the primer vector must be tangent to the unit circle at two distinct interior points. This will be referred to as the tangency condition.

Because the primer vector curve and the unit circle are symmetric about a vertical axis, the analysis is simplified. Another necessary condition for four impulses that follows from Eqs. (22) and (23) and symmetry is

$$p_1(\tau_1) = -p_1(\tau_4) \tag{24a}$$

$$p_2(\tau_1) = p_2(\tau_4) \tag{24b}$$

$$p_1(\tau_2) = -p_1(\tau_3) \tag{24c}$$

$$p_2(\tau_2) = p_2(\tau_3) \tag{24d}$$

$$p_1'(\tau_2) = p_1'(\tau_3) \tag{24e}$$

$$p_2'(\tau_2) = -p_2'(\tau_3) \tag{24f}$$

This will be called the symmetry condition.

Because of the form of the primer vector (21), another necessary condition for four-impulse solutions that follows from the inequality (12) is

$$|\boldsymbol{p}(\tau)| \le 1, \qquad \tau_1 \le \tau \le \tau_4 \tag{25}$$

stating that $\tau_1$ and $\tau_4$ must be endpoints unless $b = 0$.

Inequality (25) and the differentiability of the function defined through the left side of the inequality (25) establish the new necessary conditions,

$$\boldsymbol{p}(\tau_1)^T\boldsymbol{p}'(\tau_1) \le 0 \tag{26a}$$

$$\boldsymbol{p}(\tau_4)^T\boldsymbol{p}'(\tau_4) \ge 0 \tag{26b}$$

These will be referred to as the puncture conditions.

Another necessary condition is that the cycloidal curve be prolate, that is, the curve described by Eq. (21) should have a closed loop. This will be called the prolate condition. It restricts the parameter $b$ as follows:

$$0 \le |b| < \tfrac{2}{3} \tag{27}$$

The closed loop becomes an ellipse if $b = 0$.

Examples of one-loop primer vectors satisfying the necessary conditions (22–27) are seen in Fig. 1, beginning with the two-impulse situation of Fig. 1a, which is a limit of four-impulse solutions, and concluding with the ellipse in Fig. 1f in which part of the locus must be retraced to obtain four impulses.

## Analysis

The preceding necessary conditions will be applied to obtain useful information about four-impulse solutions. It will be assumed that the velocity increments are applied at $\tau_1$, $\tau_2$, $\tau_3$, and $\tau_4$.

### *Symmetry Arguments*

The symmetry condition is applied first on the primer vector equations (21). A more detailed mathematical development is presented here than is found in Prussing's paper[14] although the results in Eqs. (35–37) are essentially the same. It will be shown here how the constant $c$ can be effectively removed from Eq. (21). Equations (24c), (24d), and (24f), respectively, yield the following three relationships:

$$2(\sin\tau_2 + \sin\tau_3) + 3b(\tau_2 + \tau_3) + 2c = 0 \tag{28a}$$

$$\cos\tau_2 = \cos\tau_3 \tag{28b}$$

$$-\sin\tau_2 = \sin\tau_3 \tag{28c}$$

Equations (28a–28c) show that

$$\tau_2 + \tau_3 = 2\pi n \tag{29a}$$

$$c = -3\pi b n \tag{29b}$$

where $n$ can be any integer.

By the insertion of Eq. (29b) into Eq. (21a), the equation representing $p_1(\tau)$, and by the transference to the new variable

$$\hat{\tau} = \tau - \pi n \tag{30}$$

it is seen that two distinct situations can occur.

If $n$ is even, then regarding the primer vector (21) as a function of $\hat{\tau}$,

$$p_1(\hat{\tau}) = \rho(2\sin\hat{\tau} + 3b\hat{\tau}) \tag{31a}$$

$$p_2(\hat{\tau}) = \rho(\cos\hat{\tau} + 2b) \tag{31b}$$

If $n$ is odd, however, the primer vector takes the form

$$p_1(\hat{\tau}) = -\rho(2\sin\hat{\tau} - 3b\hat{\tau}) \tag{32a}$$

$$p_2(\hat{\tau}) = -\rho(\cos\hat{\tau} - 2b) \tag{32b}$$

It is observed that the form of Eq. (31) effectively allows one to set $n = 0$ in Eq. (29).

By the taking of additional advantage of symmetry, one can put Eqs. (32) into the form of Eq. (31). By the defining of the new expressions

$$\hat{b} = -b \tag{33a}$$

$$\hat{\boldsymbol{p}}(\hat{\tau}) = -\boldsymbol{p}(\hat{\tau}) \tag{33b}$$

Eqs. (32) become

$$\hat{p}_1(\hat{\tau}) = \rho(2\sin\hat{\tau} + 3\hat{b}\hat{\tau}) \tag{34a}$$

$$\hat{p}_2(\hat{\tau}) = \rho(\cos\hat{\tau} + 2\hat{b}) \tag{34b}$$

When the new notation is dropped, it is apparent that one can regard the primer vector as dependent on only two parameters, $\rho$ designating scale and $b$ determining the shape:

$$p_1(\tau) = \rho(2\sin\tau + 3b\tau) \tag{35a}$$

$$p_2(\tau) = \rho(\cos\tau + 2b) \tag{35b}$$

In view of Eqs. (29a) and (30), the two interior points of application of impulses are related by

$$\tau_2 = -\tau_3 \tag{36}$$

The curve defined by Eq. (35) is symmetric about a vertical line through the origin. For this reason the symmetry conditions (24a), (24b), and (35) establish that

$$\tau_1 = -\tau_4 \tag{37}$$

### *Tangency Arguments*

Next the tangency condition (23) is applied for two interior points using Eqs. (35). Multiplying by $1/\rho^2$, the tangency condition provides

$$(2\sin\tau_i + 3b\tau_i)(2\cos\tau_i + 3b) - (\cos\tau_i + 2b)\sin\tau_i = 0 \qquad i = 2, 3 \tag{38}$$

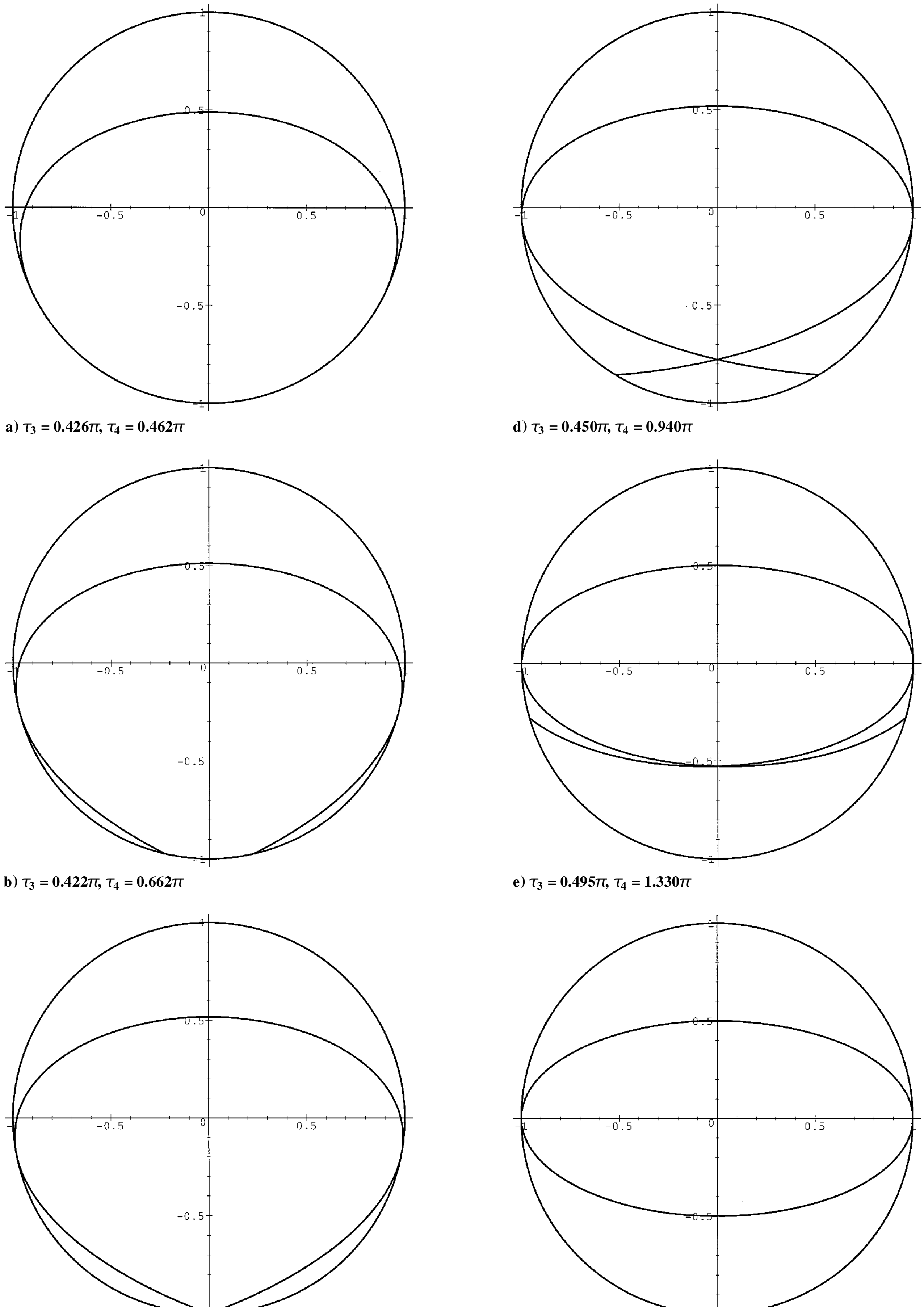
For four-impulse solutions, it is necessary that two distinct interior points, $\tau_2$ and $\tau_3$, of the application of impulses satisfy Eq. (38). By the multiplying and the rearranging of terms, these equations are quadratic in $b$,

$$b^2 + 2\beta(\tau_i)b + \gamma(\tau_i) = 0, \qquad i = 2, 3 \tag{39}$$

where

$$\beta(\tau_i) = \tfrac{2}{9}(\sin\tau_i/\tau_i) + (\cos\tau_i/3), \qquad i = 2, 3 \tag{40}$$

$$\gamma(\tau_i) = (\sin\tau_i/\tau_i)(\cos\tau_i/3), \qquad i = 2, 3 \tag{41}$$




**a) $\tau_3 = 0.426\pi$, $\tau_4 = 0.462\pi$**

**b) $\tau_3 = 0.422\pi$, $\tau_4 = 0.662\pi$**

**c) $\tau_3 = 0.424\pi$, $\tau_4 = 0.729\pi$**

**d) $\tau_3 = 0.450\pi$, $\tau_4 = 0.940\pi$**

**e) $\tau_3 = 0.495\pi$, $\tau_4 = 1.330\pi$**

**f) $\tau_3 = 0.500\pi$, $\tau_4 = 1.50\pi$**

**Fig. 1 Primer vector and unit circle for various values of $\tau_4$.**

In accord with the quadratic formula,

$$b = -\beta(\tau_i) \pm \sqrt{\beta(\tau_i)^2 - \gamma(\tau_i)}, \qquad i = 2, 3 \tag{42}$$

If a velocity impulse occurs at $\tau_i$ on the interior of the interval $[\tau_0, \tau_f]$ it is necessary that the discrimimant in Eq. (42) be nonnegative, that is,

$$\beta(\tau_i)^2 - \gamma(\tau_i) = [(\sin\tau_i/\tau_i) - 3\cos\tau_i][4(\sin\tau_i/\tau_i) - 3\cos\tau_i] \geq 0 \tag{43}$$

The smallest positive number satisfying the inequality (43) will be denoted $\mu_1$. It is the smallest positive root of the equation

$$(\sin\tau/\tau) - 3\cos\tau = 0 \tag{44}$$

An approximate value of $\mu_1$ is $0.4215\pi$. For four-impulse solutions to exist, it is necessary that $\tau_3 \geq \mu_1$.

These remarks lead to the following. It is impossible to have a four-impulse solution in which $\tau_2 > -\mu_1$ or $\tau_3 < \mu_1$. As a result, there are no four-impulse solutions if $-\mu_1 \leq \tau_0$ or $\tau_f \leq \mu_1$.

*Puncture Conditions*

The application of the puncture conditions (26) is similar to the application of the tangency conditions because the forms of the left sides of Eqs. (23) and the inequalities (26) are the same. At the points $\tau_1$ and $\tau_4$, the inequalities (26a) and (26b), respectively, become

$$b^2 + 2\beta(\tau_1)b + \gamma(\tau_1) \leq 0 \tag{45a}$$

$$b^2 + 2\beta(\tau_4)b + \gamma(\tau_4) \geq 0 \tag{45b}$$

*Intersection Conditions*

Evaluating the primer vector (35), respectively, at $\tau_3$ and $\tau_4$ and applying the intersection conditions (22) result in

$$(2\sin\tau_i + 3b\tau_i)^2 + (\cos\tau_i + 2b)^2 = \rho^{-2}, \qquad i = 3, 4 \tag{46}$$

By the elimination of $\rho^{-2}$ by subtracting the two equations, then by expansion, a quadratic equation emerges

$$b^2 + 2\delta(\tau_3, \tau_4)b + \eta(\tau_3, \tau_4) = 0 \tag{47}$$

where

$$\delta(\tau_3, \tau_4) = \frac{2}{9}\left(\frac{3\tau_3\sin\tau_3 + \cos\tau_3 - 3\tau_4\sin\tau_4 - \cos\tau_4}{\tau_3^2 - \tau_4^2}\right) \tag{48}$$

$$\eta(\tau_3, \tau_4) = \frac{\sin^2\tau_3 - \sin^2\tau_4}{3(\tau_3^2 - \tau_4^2)} \tag{49}$$

*Computation of $\tau_3$ and $\tau_4$*

There are now enough conditions to determine $\tau_3$ from $\tau_4$ or vice versa. Equations (39–41) using $i = 3$, Eqs. (47–49), and the inequality (45b) are enough, although the use of roots of a quadratic may yield two solution branches. For each branch there is the additional necessary condition: Given a number $\tau_3$, it is necessary that $\tau_4$ be the smallest solution of this system greater than $\tau_3$; if $\tau_4$ is specified, it is necessary that $\tau_3$ be the largest solution of this system less than $\tau_4$.

A solution of the system will refer to a pair $\tau_3$ and $\tau_4$ that satisfies this necessary condition, and Eqs. (39–41) with $i = 3$, Eqs. (47–49) and the inequality (45b). For this reason there can be at most two solutions of the system for a specified value of $\tau_4$, or at most two solutions of the system for a specified value of $\tau_3$.

Two computational approaches are presented. The first approach is the computation of the third-impulse point $\tau_3$ from the fourth-impulse point $\tau_4$. A mission planner picking a flight interval $[-\tau_f, \tau_f]$ will be interested in the determination of $\tau_3$ and should prefer this approach. In the second approach, $\tau_3$ is specified and $\tau_4$ is determined. The calculations are easier in the second approach, and it is more useful to graph the relation between $\tau_3$ and $\tau_4$.

*Approach 1.* Knowing the length of the flight interval, the number $\tau_4$ is specified. The two roots of Eq. (47) then depend on $\tau_3$,

$$b_1(\tau_3, \tau_4) = -\delta(\tau_3, \tau_4) + \sqrt{\delta(\tau_3, \tau_4)^2 - \eta(\tau_3, \tau_4)} \tag{50a}$$

$$b_2(\tau_3, \tau_4) = -\delta(\tau_3, \tau_4) - \sqrt{\delta(\tau_3, \tau_4)^2 - \eta(\tau_3, \tau_4)} \tag{50b}$$

Using $i = 3$ and substituting into Eq. (39) give

$$b_1(\tau_3, \tau_4)^2 + 2\beta(\tau_3)b_1(\tau_3, \tau_4) + \gamma(\tau_3) = 0 \tag{51a}$$

$$b_2(\tau_3, \tau_4)^2 + 2\beta(\tau_3)b_2(\tau_3, \tau_4) + \gamma(\tau_3) = 0 \tag{51b}$$

Each of these is solved iteratively for $\tau_3$ using only the largest root less than $\tau_4$. The root $\tau_{31}$ of Eq. (51a) is substituted via Eqs. (48) and (49) into Eq. (50a) to determine $b_1(\tau_{31}, \tau_4)$. Similarly $b_2(\tau_{32}, \tau_4)$ is calculated from Eq. (50b). Each of these two numbers is subjected to the test (45b). If one of these fails this test, then the remaining one satisfying Eq. (45b) is a unique solution of the system. If both pass, there are two solutions; if both fail, there are no solutions.

*Approach 2.* One can select $\tau_3$ and calculate $b_1(\tau_3)$ and $b_2(\tau_3)$ from Eq. (42) via Eqs. (40) and (41). By using Eqs. (48) and (49), $b_1(\tau_3)$ and $b_2(\tau_3)$ are used in Eq. (47)

$$b_1(\tau_3)^2 + 2\delta(\tau_3, \tau_4)b_1(\tau_3) + \eta(\tau_3, \tau_4) = 0 \tag{52a}$$

$$b_2(\tau_3)^2 + 2\delta(\tau_3, \tau_4)b_2(\tau_3) + \eta(\tau_3, \tau_4) = 0 \tag{52b}$$

Each of these is solved iteratively for $\tau_4$ using only the smallest root greater than $\tau_3$. Each of the two roots $\tau_{41}$ and $\tau_{42}$, respectively, are subjected to the respective tests from the inequality (45b)

$$b_1(\tau_3)^2 + 2\beta(\tau_{41})b_1(\tau_3) + \gamma(\tau_{41}) \geq 0 \tag{53a}$$

$$b_2(\tau_3)^2 + 2\beta(\tau_{42})b_2(\tau_3) + \gamma(\tau_{42}) \geq 0 \tag{53b}$$

Calculations from approach 2 were performed using a MAPLE software package, and the results are presented in Fig. 2, which consists of three parts demonstrating three ranges of values of $\tau_3$ and $\tau_4$. The straight lines show the relationship between the left ends of the component curves and are not part of the data. Figure 2 is in complete agreement with a similar figure in Prussing's[14] paper, although the shape is different. This material could be useful to a mission planner, who, given the final impulse point $\tau_4$, could estimate the approximate point of an interior impulse $\tau_3$. Six points taken from this curve were used for the primer vector plots in Fig. 1. Figure 1a shows approximately the limiting case where $\tau_3 = \tau_4$. Figure 1b shows the approximate primer vector locus associated with $\mu_1$, the minimum admissible value of $\tau_3$. Figure 1f shows the primer vector locus associated with $\tau_4 = 3\pi/2$, which is the largest value of $\tau_4$ having a single loop in the primer vector, although the cycloidal shape has degenerated to an ellipse. Elliptic primer vector shapes are the only situations in which $\tau_4$ need not be equal to $\tau_f$. Figure 3 shows six multiloop primer vector loci for larger values that appear as $\tau_4$ increases beyond $3\pi/2$. In general, Figs. 1 and 3 show the various shapes taken by the locus of the primer vector in sequence as $\tau_4$ increases.

**Boundary Values in Four-Impulse Rendezvous**

The mission planner selects the true anomaly interval $[\theta_0, \theta_f]$, the initial state vector $\boldsymbol{y}_0$, and the terminal state vector $\boldsymbol{y}_f$ for the rendezvous maneuver. The boundary point $z_f$ that is used in (2) is obtained[25] from

$$z_f = \Phi(\theta_f)^{-1}\boldsymbol{y}_f - \Phi(\theta_0)^{-1}\boldsymbol{y}_0 \tag{54}$$

It is convenient to transform the interval $\theta_0 \leq \theta \leq \theta_f$ to a new interval $\theta_0^* \leq \theta^* \leq \theta_f^*$ by the change of variable

$$\theta^* = \theta - [(\theta_f + \theta_0)/2] \tag{55}$$

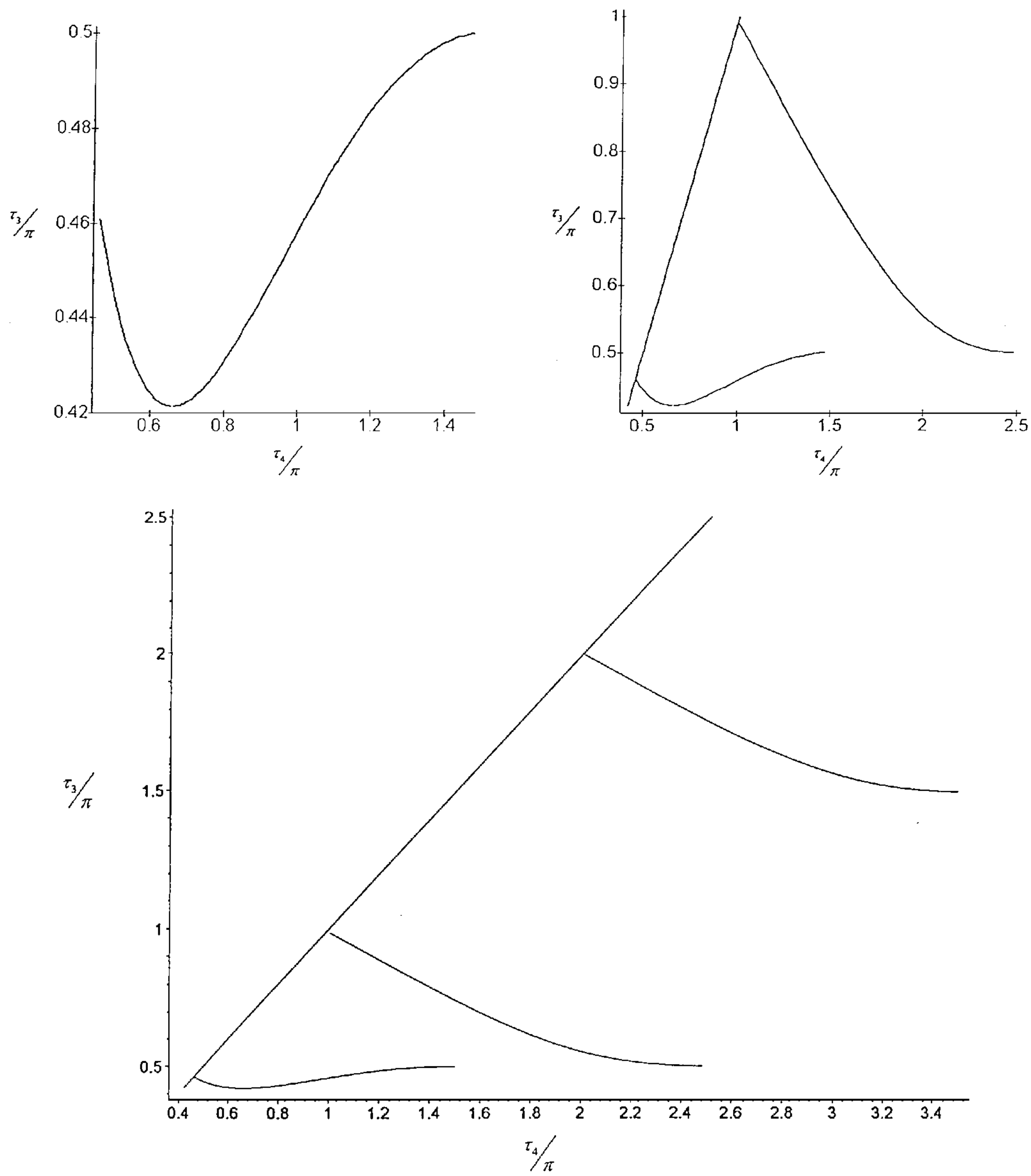



**Fig. 2 Relation between points of application of third and fourth impulse.**

The endpoints of the transformed interval are

$$\theta_0^* = -[(\theta_f - \theta_0)/2], \qquad \theta_f^* = (\theta_f - \theta_0)/2 \tag{56}$$

so that $\theta_0^* = -\theta_f^*$.

The necessary conditions (7–12) and the subsequent work that followed from them will be applied using the variable $\theta^*$ over the interval $[\theta_0^*, \theta_f^*]$.

*Boundary Points*

When the new variable $\tau = \theta^* - \varphi$ is introduced through Eq. (19), the preceding symmetry arguments ultimately show through Eq. (37) that $\tau_0 = -\tau_f$ unless $b = 0$ and $\tau_f > \tau_4$ or $\tau_0 < \tau_1$. Except for that abnormality, Eq. (54) is modified to become

$$z_f = \Phi(\tau_4)^{-1}\mathbf{y}_f - \Phi(\tau_1)^{-1}\mathbf{y}_0 \tag{57}$$

From Eq. (19), $\varphi = \theta_f^* - \tau_f = \theta_0^* - \tau_0$ and so if $b \neq 0$, then $\tau = \theta^*$ because

$$\varphi = \tfrac{1}{2}\left[(\theta_0^* + \theta_f^*) - (\tau_0 + \tau_f)\right] = 0 \tag{58}$$

If $b = 0$, then $\varphi = -(\tau_0 + \tau_f)$, but $\tau_3 = \pi/2$ and $\tau_4$ is an odd multiple of $\pi/2$. In this aberrant case, $\tau_0$ and $\tau_f$ are not unique.

The remainder of the analysis assumes that either $\tau_3 \neq \pi/2$ or else $\tau_0 = \tau_1$ and $\tau_f = \tau_4$. Boundary condition (8), thus, becomes

$$\sum_{i=1}^{4} R(\tau_i)\boldsymbol{p}(\tau_i)\alpha_i = -z_f \tag{59}$$

where $z_f$ is now determined from Eq. (57).

By the use of Eqs. (3), (14), (15), and (35), expression (59) becomes

$$\sum_{I=1}^{4} \boldsymbol{f}(\tau_i)\alpha_i = -\frac{z_f}{\rho} \tag{60}$$

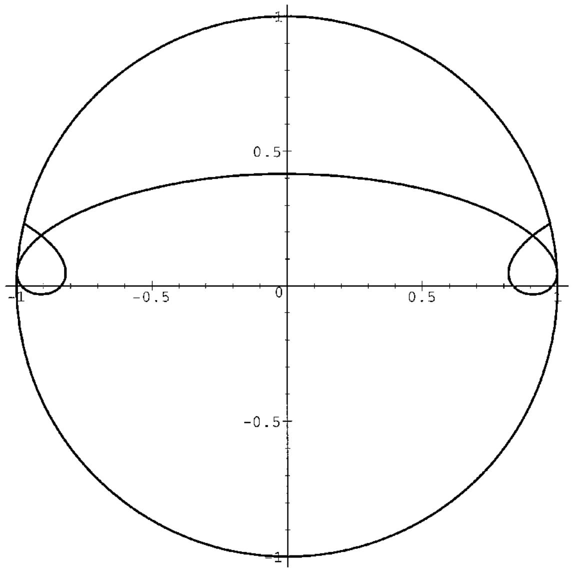
where $\boldsymbol{f}(\tau) = [f_1(\tau), f_2(\tau), f_3(\tau), f_4(\tau)]^T$ is in four-dimensional Euclidean space for each $\tau \in [\tau_0, \tau_f]$ and

$$f_1(\tau) = 1 + 3\sin^2\tau + 2(\cos\tau + 3\tau\sin\tau)b \tag{61a}$$

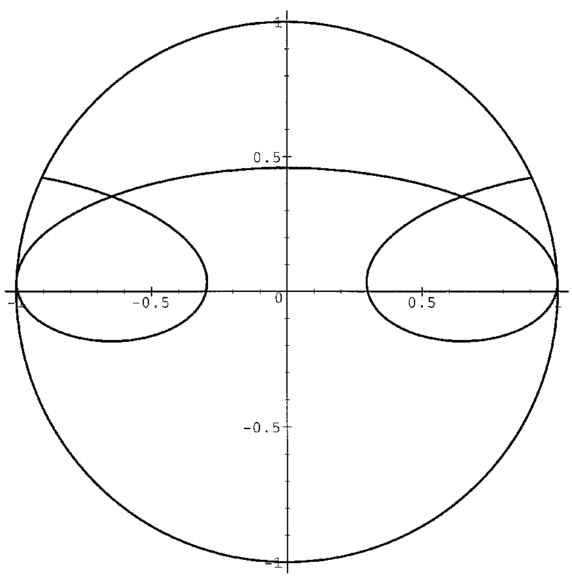
$$f_2(\tau) = -3\sin\tau\cos\tau + 2(\sin\tau - 3\tau\cos\tau)b \tag{61b}$$

$$f_3(\tau) = 2\sin\tau + 3\tau b \tag{61c}$$

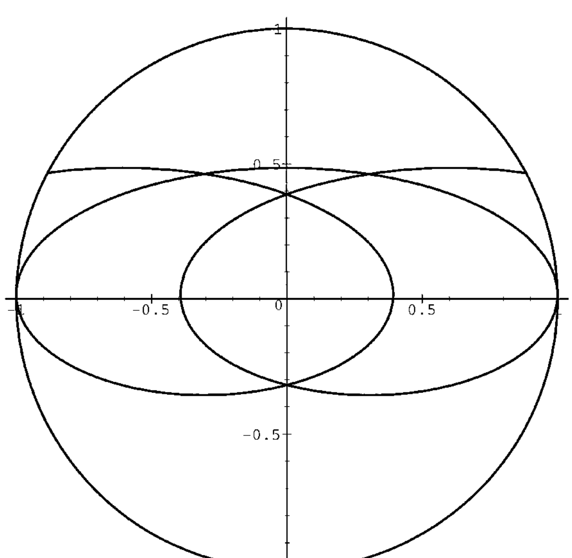
$$f_4(\tau) = 2(\cos\tau + 3\tau\sin\tau) + (4 + 9\tau^2)b \tag{61d}$$



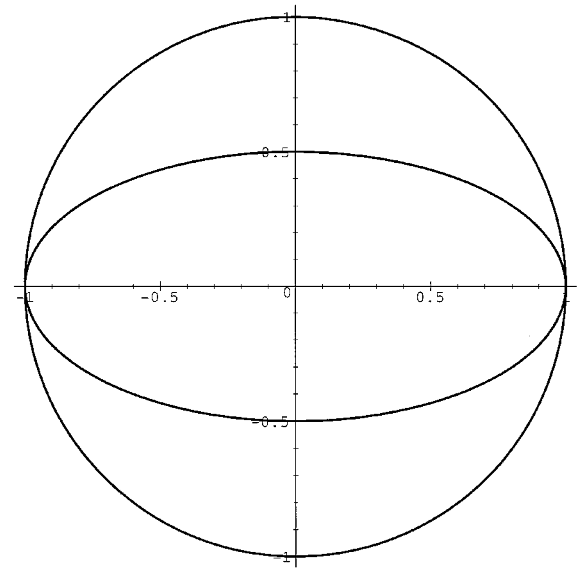

a) $\tau_3 = 0.700\pi$, $\tau_4 = 1.59\pi$



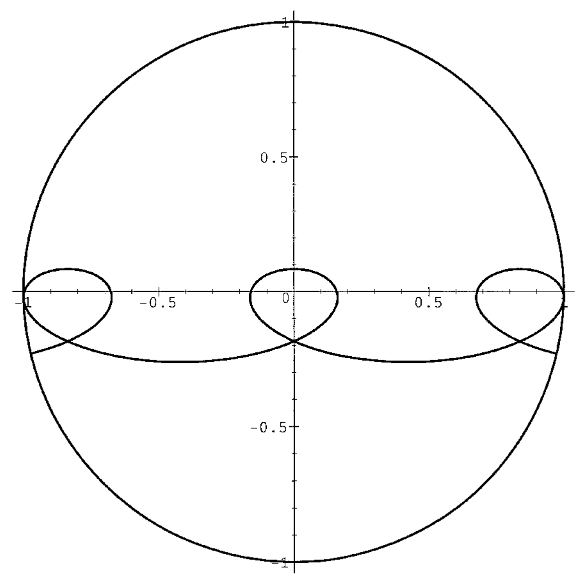

b) $\tau_3 = 0.589\pi$, $\tau_4 = 1.87\pi$



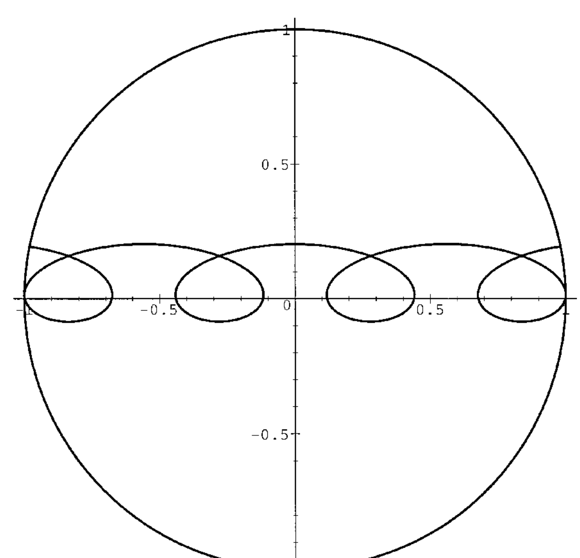

c) $\tau_3 = 0.530\pi$, $\tau_4 = 2.11\pi$




d) $\tau_3 = 0.500\pi$, $\tau_4 = 2.50\pi$




e) $\tau_3 = 1.63\pi$, $\tau_4 = 2.77\pi$




f) $\tau_3 = 2.59\pi$, $\tau_4 = 3.91\pi$

**Fig. 3 Primer vector and unit circle for various values of $\tau_4 > 1.5\pi$.**

Condition (59) can be written in terms of a matrix $M$ whose columns, respectively, are $f(\tau_1)$, $f(\tau_2)$, $f(\tau_3)$, and $f(\tau_4)$ and a vector $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$:

$$M\alpha = -z_f/\rho \tag{62}$$

Because $\tau_1 = -\tau_4$ and $\tau_2 = -\tau_3$, this matrix depends only on $\tau_3$ and $\tau_4$, which are related as shown in Fig. 2,

$$M = \begin{bmatrix} f_1(\tau_4) & f_1(\tau_3) & f_1(\tau_3) & f_1(\tau_4) \\ -f_2(\tau_4) & -f_2(\tau_3) & f_2(\tau_3) & f_2(\tau_4) \\ -f_3(\tau_4) & -f_3(\tau_3) & f_3(\tau_3) & f_3(\tau_4) \\ f_4(\tau_4) & f_4(\tau_3) & f_4(\tau_3) & f_4(\tau_4) \end{bmatrix} \tag{63}$$

It is known[24] that the columns of $M$ are linearly independent, and so $M$ is invertible and

$$\alpha = -M^{-1}z_f/\rho \tag{64}$$

Recall the inequalities (9); this establishes the following result.

*Theorem:* Suppose $\tau_0 = -\tau_f$, $\tau_1 = \tau_0$, $\tau_2 = -\tau_3$, and $\tau_4 = -\tau_1$ and $\tau_3$ and $\tau_4$ are solutions of the system. Then a boundary point $\boldsymbol{z}_f$ admits a nondegenerate four-impulse solution if and only if each component of the vector $M^{-1}\boldsymbol{z}_f$ is negative. If $\boldsymbol{z}_f$ has a nondegenerate four-impulse solution, then the velocity impulses are

$$\Delta v_i = -p(\tau_i)\alpha_i, \qquad i = 1, 2, 3, 4$$

*Remark*: Nondegenerate solutions are defined in previous work.[25]

*Example*

A spacecraft has its velocity synchronized with a satellite in circular orbit. Its purpose is to rendezvous with the satellite in exactly one orbital revolution. It will be shown that there are no four-impulse solutions to this problem.

For this problem, the initial coordinates of the spacecraft relative to the satellite will be denoted, respectively, by $y_{10}$ and $y_{20}$. Because one orbital revolution consists of an angle of $2\pi$ rad, one sets $\tau_0 = -\pi$ and $\tau_f = \pi$. The terminal value of the state vector is centered in the satellite, which is the origin of the coordinate system, so that $\boldsymbol{y}_f = 0$. The boundary point (54) becomes

$$z_f = -\Phi(-\pi)^{-1}\boldsymbol{y}_0 \tag{65}$$

The initial velocity of the spacecraft relative to the satellite is zero, so that the initial state vector becomes

$$\boldsymbol{y}_0 = (y_{10}, 0, y_{20}, 0)^T \tag{66}$$

The interior impulse point $\tau_3$ can be approximated from Fig. 2 or determined more accurately from the system (39–41), (45b), and (47–49). Calculation from the system yields $\tau_3 = 0.459\pi$. The four-impulse points are, therefore, known because $\tau_2 = -\tau_3$ and $\tau_0 = \tau_1 = -\tau_4 = -\tau_f$. From the system, one obtains $b = -0.1017$. By the use of Eq. (46), one calculates $\rho = 0.64856$.

The matrix $M$ is then calculated from Eqs. (61) and (63). According to the theorem, it is necessary for each component of the vector $M^{-1}z_f$ to be negative. This results in the inequalities

$$-1.48950y_{20} - 0.0356778y_{10} < 0 \tag{67a}$$

$$0.405727y_{20} + 0.0140688y_{10} < 0 \tag{67b}$$

$$0.124655y_{20} + 0.0140688y_{10} < 0 \tag{67c}$$

$$0.144479y_{20} - 0.0356778y_{10} < 0 \tag{67d}$$

This system of inequalities has no solution. This proves that there is no four-impulse optimal solution to this rendezvous problem.

If one removes the restriction that the spacecraft velocity is synchronized with that of the satellite, then there are many initial conditions that admit four-impulse solutions.

## Mission-Design Procedures for Rendezvous Maneuvers that Require Four Impulses

The mission designer selects the initial instant and state $(\theta_0, \boldsymbol{y}_0)$ and the final instant and state $(\theta_f, \boldsymbol{y}_f)$.

1) If $\theta_f - \theta_0 \leq 0.924\pi$, then stop. There can be no four-impulse rendezvous regardless of the boundary conditions. Otherwise, calculate $\tau_4 = \frac{1}{2}(\theta_f - \theta_0)$, and apply approach 1 as follows.

2) Calculate $\tau_{31}$, the largest root of Eq. (51a) that is less than $\tau_4$. Similarly calculate $\tau_{32}$, the largest root of Eq. (51b) that is less than $\tau_4$.

3) By the use of $\tau_4$ and $\tau_{31}$, calculate $b_1(\tau_{31}, \tau_4)$ from Eq. (50a); by the use of $\tau_4$ and $\tau_{32}$, calculate $b_2(\tau_{32}, \tau_4)$ from Eq. (50b).

4) Check the necessary condition (45b) setting $b = b_1(\tau_{31}, \tau_4)$. Repeat setting $b = b_2(\tau_{32}, \tau_4)$. If both violate this condition, then stop. There can be no four-impulse rendezvous.

5) If $b_1(\tau_{31}, \tau_4)$ satisfies condition in Eq. (45b), calculate the functions $f_1(\tau_{31})$, $f_2(\tau_{31})$, $f_3(\tau_{31})$, and $f_4(\tau_{31})$ and $f_1(\tau_4)$, $f_2(\tau_4)$, $f_3(\tau_4)$, and $f_4(\tau_4)$ from Eq. (61) using $b = b_1(\tau_{31}, \tau_4)$. If $b_2(\tau_{32}, \tau_4)$ satisfies condition in Eq. (45b), calculate $f_1(\tau_{32})$, $f_2(\tau_{32})$, $f_3(\tau_{32})$, and $f_4(\tau_{32})$ and $f_1(\tau_4)$, $f_2(\tau_4)$, $f_3(\tau_4)$, and $f_4(\tau_4)$ from Eq. (61) using $b = b_2(\tau_{32}, \tau_4)$.

6) For either or both cases satisfying condition in Eq. (45b), calculate the matrix $M$ from Eq. (63) using $\tau_3 = \tau_{31}$ and $b_1(\tau_{31}, \tau_4)$ or $\tau_3 = \tau_{32}$ and $b_2(\tau_{32}, \tau_4)$ or both as applies, then invert this matrix to obtain $M^{-1}$.

7) Based on Eq. (54), the boundary value is calculated from

$$z_f = \Phi(\tau_4)^{-1}\boldsymbol{y}_f - \Phi(-\tau_4)^{-1}\boldsymbol{y}_0$$

where $\Phi^{-1}$ is obtained from Eq. (14).

8) For either or both cases satisfying condition in Eq. (45b), calculate the vector $M^{-1}\boldsymbol{z}_f$. If any component of this vector is not negetive, that case does not admit nondegenerate four-impulse solutions. If this holds for all cases satisfying condition in Eq. (45b), then stop. There are nondegenerate four-impulse solutions.

9) At most two cases satisfy condition in Eq. (45b). If $b_1(\tau_{31}, \tau_4)$ satisfies condition in Eq. (45b) and all components of $M^{-1}\boldsymbol{z}_f$ are negative, calculate $\rho^{-2}$ from Eq. (46) using $b = b_1(\tau_{31}, \tau_4)$ and $\tau_i = \tau_4$. Invert this number and take the positive square root to obtain $\rho$. If $b_2(\tau_{32}, \tau_4)$ satisfies condition in Eq. (45b) and all components of $M^{-1}\boldsymbol{z}_f$ are negative, calculate $\rho^{-2}$ similarly from Eq. (46) using $b = b_2(\tau_{32}, \tau_4)$ and obtain $\rho$. For each or both cases as applies, calculate $\alpha$ from Eq. (63). The velocity increments are calculated from

$$\Delta v_i = -p(\tau_i)\alpha_i, \qquad i = 1, 2, 3, 4$$

where $\tau_1 = -\tau_4$, $\tau_2 = -\tau_3$, and $p(\tau_i)$ is calculated from Eq. (35) calculating $\tau_3$, $b$, and $\rho$ from $\tau_{31}$ or $\tau_{32}$, or both as the case applies.

## Conclusions

The basic mathematical tools are established for a thorough understanding of the conditions that result in four-impulse solutions for the problem of optimal impulsive rendezvous of a spacecraft in the vicinity of a circular orbit. As a result, algorithms for calculation of four-impulse solutions were presented, based on the quadratic formula. Two approaches were found. The first approach allows one to calculate the four optimal velocity increments from the boundary conditions and a specified flight interval. The second approach specifies the time of the third impulse and allows the calculation of the four optimal velocity increments and the flight interval from the boundary conditions.

## References

[1] Edelbaum, T. N., "Minimum-Impulse Transfers in the Near Vicinity of a Circular Orbit," *Journal of Astronautical Sciences*, Vol. 14, No. 2, 1967, pp. 66–73.

[2] Jones, J. B., "Optimal Rendezvous in the Neighborhood of a Circular Orbit," *Journal of Astronautical Sciences*, Vol. 24, No. 1, 1976, pp. 53–90.

[3] Wheelon, A. D., "Midcourse and Terminal Guidance," Space Technology, Wiley, New York, 1959, pp. 26-28–26-32.

[4] Clohessy, W. H., and Wiltshire, R. S., "Terminal Guidance System for Satellite Rendezvous," *Journal of Aerospace Sciences*, Vol. 27, No. 9, 1960, pp. 653–658, 674.

[5]Geyling, F. T., "Satellite Perturbations from Extra-Terrestrial Gravitation and Radiation Pressure," *Journal of the Franklin Institute*, Vol. 269, No. 5. 1960, pp. 375–407.
[6]Spradlin, L. W., "The Long-Time Satellite Rendezvous Trajectory," *Aerospace Engineering*, Vol. 19, June 1960, pp. 32–37.
[7]Eggleston, J. M., "Optimum Time to Rendezvous," *ARS Journal*, Vol. 30, Nov. 1960, pp. 1089–1091.
[8]Chobotov, V. A. (ed.), *Orbital Mechanics*, AIAA Education Series, AIAA, Washington, DC, 1991, pp. 175–178.
[9]Prussing, J. A., and Conway, B. A., *Orbital Mechanics*, Oxford Univ. Press, New York, 1993, pp. 142–153.
[10]Wiesel, W. E., *Spaceflight Dynamics*, McGraw-Hill, New York, 1997, pp. 80–85.
[11]Lawden, D. F., *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963, pp. 56–68.
[12]Lion, P. M., *A Primer on the Primer*, Dept. of Aerospace and Mechanical Sciences, STAR Memo 1, Princeton Univ., Princeton, NJ, 1967.
[13]Lion, P. M., and Handelsman, M., "Primer Vector on Fixed-Time Impulsive Trajectories," *AIAA Journal*, Vol. 6, No. 1, 1968, pp. 127–132.
[14]Prussing, J. E., "Optimal Four-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit," *AIAA Journal*, Vol. 7, No. 5, 1969, pp. 928–935.
[15]Prussing, J. E., "Optimal Two and Three-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit," *AIAA Journal*, Vol. 8, No. 7, 1970, pp. 1221–1228.
[16]Jezewski, D. J., and Donaldson, J. D., "An Analytic Approach to Optimal Rendezvous Using Clohessy–Wiltshire Equations," *Journal of Astronautical Sciences*, Vol. 27, No. 3, 1979, pp. 293–310.
[17]Jezewski, D. J., "Primer Vector Theory Applied to the Linear Relative-Motion Equations," *Optimal Control Applications and Methods*, Vol. 1, 1980, pp. 387–401.
[18]Neff, J. M., and Fowler, W. T., "Minimum-Fuel Rescue Trajectories for the Extravehicular Excursion Unit," *Journal of Astronautical Sciences*, Vol. 39, No. 1, 1991, pp. 21–45.
[19]Kechichian, J. A., "Techniques of Accurate Analytic Terminal Rendezvous in Near-Circular Orbit," *Acta Astronautica*, Vol. 26, No. 6, 1992, pp. 377–394.
[20]Prussing, J. E., and Clifton, R. S., "Optimal Multiple-Impulse, Satellite Evasive Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 3, 1994, pp. 599–606.
[21]Lopez, I., and McInnes, C. R., "Autonomous Rendezvous Using Artificial Potential Function Guidance," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 2, 1995, pp. 237–241.
[22]Neustadt, L. W., "Optimization, a Moment Problem, and Nonlinear Programming," *SIAM Journal on Control*, Vol. 2, No. 1, 1964, pp. 33–53.
[23]Stern, R. G., and Potter, J. E., "Optimization of Midcourse Velocity Corrections," *Peaceful Uses of Automation in Outer Space*, Plenum, New York, 1966, pp. 70–83.
[24]Carter, T. E., "Optimal Impulsive Space Trajectories Based on Linear Equations," *Journal of Optimization Theory and Applications*, Vol. 70, No. 2, 1991, pp. 277–297.
[25]Carter, T. E., and Brient, J., "Linearized Impulsive Rendezvous Problem," *Journal of Optimization Theory and Applications*, Vol. 86, No. 3, 1995, pp. 553–584.
[26]Prussing, J. E., "Optimal Impulsive Linear Systems: Sufficient Conditions and Maximum Number of Impulses," *Journal of Astronautical Sciences*, Vol. 43, No. 2, 1995, pp. 195–206.
[27]Carter, T. E., "State Transition Matrices for Terminal Rendezvous Studies: Brief Survey and New Example," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, 1998, pp. 148–155.